§4.2. The Definite Integral.

- The Definite Integral of $f(x)$ from $x=a$ to $x=b$ is denoted by

$$\int_a^b f(x)\,dx := \text{"Area under the curve"} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\cdot\Delta x, \quad \Delta x = \frac{b-a}{n}, \quad x_i = a + i\cdot\frac{b-a}{n}.$$

Remark: $\int$: integral notation a: lower limit. b: upper limit. dx: integral w.r.t. x variable

Remark: Integral "=" Area under the curve, only depends on f and a, b (is a number)

The variable dx can be changed to any other variable. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\cdot du$

- Give a Riemann Sum, how to find the corresponding Integral $\int_a^b f(x)\,dx$?

eg.1. The limit $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{7+\frac{4i}{n}} \cdot \frac{4}{n}$ is the limit of a Riemann Sum for a certain definite integral. $\int_a^b f(x)\,dx$. Find the exact form of $\int_a^b f(x)\,dx$.

Solution: $\frac{4}{n}$ plays the role of $\Delta x = \frac{b-a}{n}$, i.e., $b-a=4$. the interval is 4 units long.

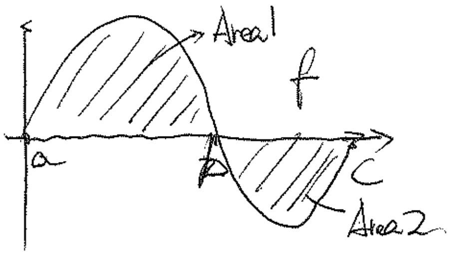
$\frac{1}{7+\frac{4i}{n}}$ plays the role of $f(x_i)$, which suggests f is $\frac{1}{\cancel{\quad}}$

Let $f(x) = \frac{1}{x}$. And our $x_1 = 7 + \frac{4}{n}$, $x_n = 7 + 4$, $x_i = 7 + i\cdot\frac{4}{n}$.

Pick $a=7$, $b=11$. Then $\int_7^{11} \frac{1}{x}\,dx$ fits the Riemann Sum.

Remark: The answer is not unique. One can check $\int_0^4 \frac{1}{7+x}\,dx$ is also correct.

- The integral represents "the area with signs". If the curve is under the x-axis, ~~the~~ the area is considered as negative.



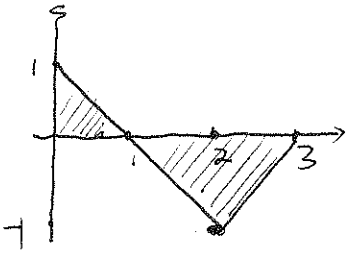
$$\int_a^b f(x)\,dx = \text{Area1}. \qquad \int_b^c f(x)\,dx = -\text{Area2}$$

$$\int_a^c f(x)\,dx = \text{Area1} - \text{Area2}.$$

eg.2. Consider the function $f(x)=\begin{cases} 1-x & 0\le x\le 2 \\ x-3 & 2\le x\le 3 \end{cases}$

Use the graph of $f(x)$ on $[0,3]$ to find $\int_0^1 f(x)dx$, $\int_1^2 f(x)dx$, $\int_0^2 f(x)dx$, $\int_1^3 f(x)dx$, $\int_0^3 f(x)dx$

Solution:

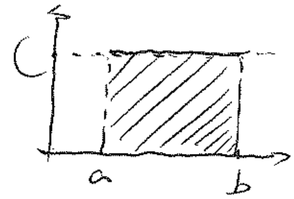


$\int_0^1 f(x)dx=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}$, $\int_1^2 f(x)dx=-\frac{1}{2}$, $\int_1^3 f(x)dx=-1$

$\int_0^2 f(x)dx=\frac{1}{2}-\frac{1}{2}=0$. $\int_0^3 f(x)dx=\frac{1}{2}-1=-\frac{1}{2}$.

- Some basic integrals from the graph:

Rectangle $\int_a^b C\,dx=C(b-a)$.

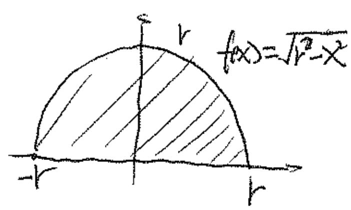


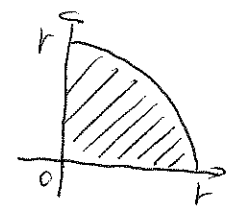
$f(x)=C$ (constant C)

Area $=C(b-a)$.

Half/Quarter Disk: $\int_{-r}^{r}\sqrt{r^2-x^2}\,dx=\frac{1}{2}\pi r^2$

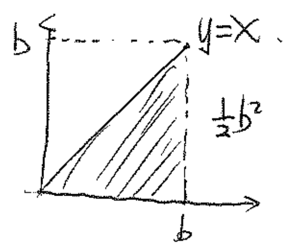
$\int_0^r\sqrt{r^2-x^2}\,dx=\frac{1}{4}\pi r^2$

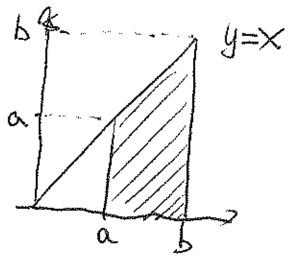




Triangle/Trapezoid: $\int_0^b x\,dx=\frac{1}{2}b^2$

$\int_a^b x\,dx=\frac{1}{2}b^2-\frac{1}{2}a^2$

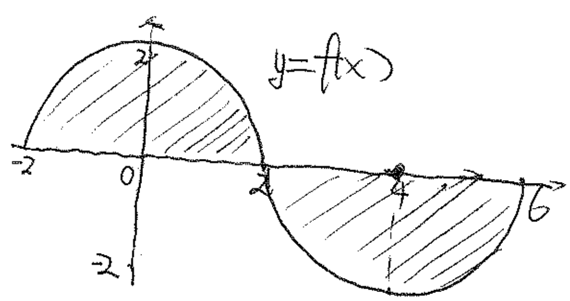




eg.3. Suppose the graph of $y=f(x)$ is as follow.

All curves are half circles with radius 2.

Find. $\int_{-2}^{2}f(x)dx$, $\int_0^2 f(x)dx$, $\int_0^4 f(x)dx$



$\int_{-2}^{2}f(x)dx=\frac{1}{2}\pi\cdot 2^2=2\pi$

$\int_0^2 f(x)dx=\frac{1}{4}\pi\cdot 2^2=\pi$

$\int_0^4 f(x)dx=\frac{1}{4}\pi\cdot 2^2-\frac{1}{4}\pi\cdot 2^2=0$

$\int_2^6 f(x)dx=-\frac{1}{2}\pi\cdot 2^2=-2\pi$

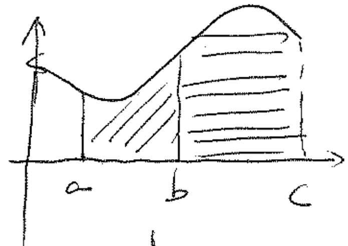
$\int_0^6 f(x)dx=\pi-(2\pi)=-\pi$. $\int_{-2}^{6}f(x)dx=2\pi-2\pi=0$

- Properties of Definite integrals

① $\int_a^a f(x)\,dx = 0$. (The area is zero if the upper and lower limits coincide).

② $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$. (Flip the lower and upper limits by adding a negative sign)

eg. $\int_5^2 2x\,dx = -\int_2^5 2x\,dx$.



☆ ③ Splitting $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$.

④ linear properties $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$; $\int_a^b C f(x)\,dx = C \int_a^b f(x)\,dx$.

☆ eg4. (#16) Suppose $\int_2^5 f(x)\,dx = 3$, $\int_2^3 f(x)\,dx = -4$. Find $\int_5^3 2f(x)\,dx$.

Solution: $\int_5^3 2f(x)\,dx = \int_5^2 2f(x)\,dx + \int_2^3 2f(x)\,dx$ splitting $\int_5^3 = \int_5^2 + \int_2^3$

$= -\int_2^5 2f(x)\,dx + \int_2^3 2f(x)\,dx$. flipping $\int_5^2 = -\int_2^5$

$= -2 \cdot \int_2^5 f(x)\,dx + 2 \cdot \int_2^3 f(x)\,dx$. constant multiple

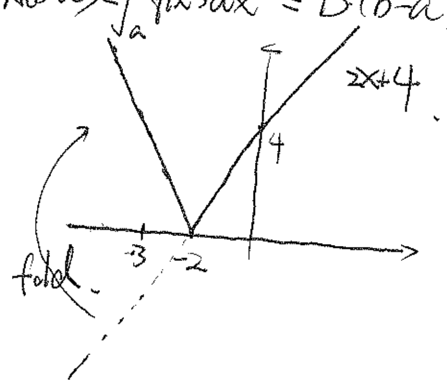
$= -2 \cdot 3 + 2 \cdot (-4) = \boxed{-14}$.

Hints for webwork:

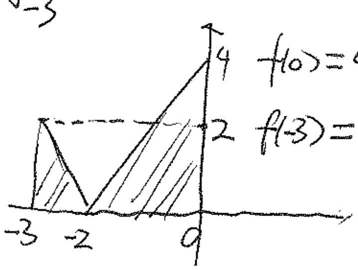
※10: ⑤ Bounds: If $A \le f(x) \le B$, then $A(b-a) \le \int_a^b f(x)\,dx \le B(b-a)$

☆ ※5. Graph of absolute value:

$y = |2x+4|$



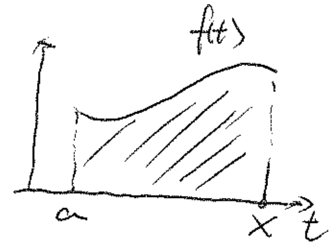
$\int_{-3}^{0} |2x+4|\,dx = \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \times 2 \times 4 = 5$



$f(0) = 4$

$f(-3) = |2 \cdot (-3) + 4| = 2$

§ Fundamental Theorem of Calculus.



Key formula: ① FTC P1: If $F(x)=\int_a^x f(t)\,dt$, then $F'(x)=f(x)$.

★ If $F(x)=\int_{V(x)}^{U(x)} f(t)\,dt$, then

$$F'(x)=f(U(x))\cdot U'(x)-f(V(x))\cdot V'(x). \quad \text{(chain rule form)}$$

$$\left(\int_a^{U(x)} f(t)\,dt\right)'=f(U(x))\cdot U'(x), \quad \left(\int_{V(x)}^{b} f(t)\,dt\right)'=-f(V(x))\cdot V'(x).$$

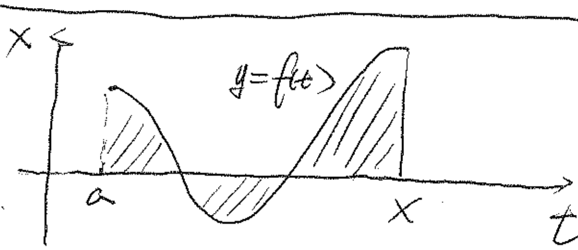
② FTC P2: If $F'(x)=f(x)$ (F is one anti-D of $f(x)$)

then $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b)-F(a)$.

③ AntiD (Integral Table):

$(n\neq -1)$: $\int_a^b x^n\,dx=\frac{1}{n+1}x^{n+1}\Big|_a^b$, $\int_a^b \cos x\,dx=\sin x\Big|_a^b$, $\int_a^b \sin x\,dx=-\cos x\Big|_a^b$

$\int_a^b \sec^2 x\,dx=\tan x\Big|_a^b$, $\int_a^b \sec x\cdot\tan x\,dx=\sec x\Big|_a^b$.



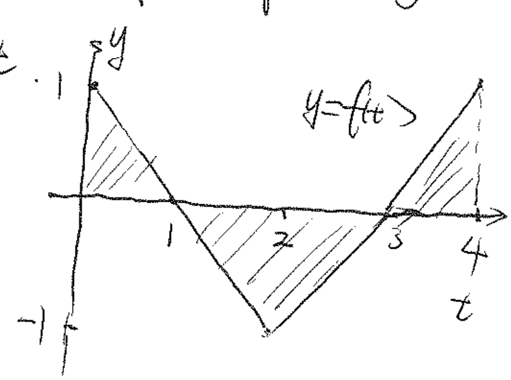
Consider $y=f(t)$ from $t=a$ to $t=x$.

$\int_a^x f(t)\,dt$ is the area of the shadow region.

It changes as x moves, therefore, is a function of the upper limit x.

We denote it by $F(x)=\int_a^x f(t)\,dt$. FTC P1, P2 tell us the relation between f and F and how to use anti D to compute a definite integral.

e.g. 1. $y=f(t)$ the graph is given below. let $g(x)=\int_0^x f(t)\,dt$.



$g(0)=\int_0^0 f(t)\,dt=0$, $g(1)=\int_0^1 f(t)\,dt=\frac{1}{2}$, $g(2)=0$

$g(3)=-\frac{1}{2}$, $g(4)=0$

★ $g(x)$ is increasing on $[0,1]\cup[3,4]$, decreasing on $[1,3]$.

- Derivative formulas:

e.g.2. Find $\frac{d}{dx}\int_{-9}^{x}(\cos t^2 + t)\,dt = \left(\int_{-9}^{x}\cos t^2 + t\ dt\right)'$

sln: Apply FTC P1 with $f(t) = \cos t^2 + t$. (Replace t in $f(t)$ by x).

$\left(\int_{-9}^{x}\cos t^2 + t\ dt\right)' = \cos x^2 + x$.

eg3. Let $h(x) = \int_{\tan x}^{3}\sqrt{2+t^2}\,dt$. Find $h'(x)$.

sln: Apply FTC P1 with $u(x)=3$ (const), $V(x) = \tan x$. $f(t) = \sqrt{2+t^2}$

$h'(x) = 0 - \sqrt{2+(\tan x)^2}\cdot(\tan x)' = -\sqrt{2+(\tan x)^2}\cdot\sec^2 x$.

↑ replace t by lower limit $\tan x$.

- Evaluate the definite integral by finding the anti-D

eg.4. Evaluate $\int_0^{\frac{\pi}{3}} 4\cdot\sec x\cdot\tan x\,dx$.

Solution: $\int_0^{\frac{\pi}{3}} \underline{4\cdot\sec x\cdot\tan x}\,dx$. Step 1: $f(x) = 4\sec x\tan x$. Find anti-D $F(x)$ of $f(x)$

$F(x) = 4\cdot\sec x$. (since $(\sec x)' = \sec x\cdot\tan x$)

Step2: FTC P2: $\int_0^{\frac{\pi}{3}} 4\sec x\cdot\tan x\,dx = 4\sec x\Big|_0^{\frac{\pi}{3}} = 4\sec\frac{\pi}{3} - 4\sec 0$. Hint: $\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = 2$

$= 4\cdot 2 - 4\cdot 1 = \boxed{4}$ $\qquad \sec 0 = \frac{1}{\cos 0} = 1$

eg.5 Evaluate $\int_1^{\sqrt{7}} \frac{13\cdot s^4 + 5\cdot\sqrt{s}}{s^4}\,ds$

Solution: $f(s) = \frac{13\cdot s^4 + 5\cdot\sqrt{s}}{s^4} = \frac{13\cdot s^4}{s^4} + \frac{5\cdot s^{\frac{1}{2}}}{s^4} = 13 + 5\cdot s^{\frac{1}{2}-4} = 13 + 5\cdot s^{-\frac{7}{2}}$

anti-D: $F(s) = 13\cdot s + 5\cdot\frac{1}{-\frac{7}{2}+1}\cdot s^{-\frac{7}{2}+1} = 13s + 5\cdot\frac{1}{-\frac{5}{2}}\cdot s^{-\frac{5}{2}}$

$= 13s - 2\cdot s^{-\frac{5}{2}}$

FTC P2 $\Rightarrow \int_1^{\sqrt{7}} \frac{13s^4 + 5\sqrt{s}}{s^4}\,ds = (13\cdot s - 2\cdot s^{-\frac{5}{2}})\Big|_1^{\sqrt{7}} = \left(13\cdot\sqrt{7} - 2(\sqrt{7})^{-\frac{5}{2}}\right) - \left(13\cdot 1 - 2\cdot 1^{-\frac{5}{2}}\right)$

$= \boxed{13\cdot\sqrt{7} - 2\cdot(\sqrt{7})^{-\frac{5}{2}} - 11}$

- $\int_a^a f(x)dx =$ ____________; (flipping) : $\int_b^a f(x)dx =$ ____________________

- splitting : $\int_a^b f(x)dx + \int_b^c f(x)dx =$ ____________________

- $\int_a^b f(x) \pm g(x)dx =$ __________ $\pm$ __________, $\int_a^b C \cdot f(x)dx =$ ____________________

- $\int_a^b 1dx =$ ____________________, $\int_a^b Cdx =$ ____________________

- If $F(x) = \int_a^x f(t)\ dt$, then $F'(x) =$.

- If $F(x) = \int_a^{u(x)} f(t)\ dt$, then $F'(x) =$ ____________________.

- If $F(x) = \int_{v(x)}^b f(t)\ dt$, then $F'(x) =$ ____________________.

- If $F(x)$ is an anti-D of $f(x)$, i.e., $F'(x) = f(x)$, then $\int_a^b f(x)\ dx =$ ____________________

- ***Antiderivative (Integral) Table:***

$f(x)$	$x^n, n \neq -1$	$\cos x$	$\sin x$	$\sec^2 x$	$\sec x \cdot \tan x$
Anti-D F(x)					
Definite Integral $\int_a^b f(x)dx$					

- Graph of $y = \frac{1}{x}$

- Graph of $y = |-2x+6|$

- Graph of $y = x^2 - 2, y = -x^2 + 1$